THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G/H University Mathematics for Applications 2014-2015
Revision 1

Note: Questions will be discussed in lectures, no typed solution will be given.

1. Suppose the curve $\mathcal{C}$ is given by the equation
$$xy^2 - x^2 = 3.$$
Show that $(3, 2)$ lies on the curve $\mathcal{C}$ and find the equations of the tangent line and the normal at that point.

2. By considering the substitution $t = \tan\dfrac{x}{2}$, find
   (a) $\displaystyle\int \frac{1}{\sin 2x - 2\sin x}\,dx$
   (b) $\displaystyle\int \frac{1}{(2+\cos x)\sin x}\,dx$

3. Evaluate the derivative of each of the following functions:
   (a) $\displaystyle f(x) = \int_0^{x^2} e^{\sin t + \cos 2t}\,dx$
   (b) $\displaystyle f(x) = \int_x^{x^2} e^{\sin t + \cos 2t}\,dx$

4. (a) Prove that $\displaystyle\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,dx.$
   (b) Hence, evaluate $\displaystyle\int_0^1 \frac{1}{(x^2 - x + 1)(e^{2x-1} + 1)}\,dx.$

5. (a) Suppose $f(x)$ is continuous on $[0, a]$. Show that
$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$
   Further, if $f(x) + f(a-x) = K$ for all $x \in [0, a]$, where K is a constant, prove that
   (i) $K = 2f(\frac{a}{2})$
   (ii) $\displaystyle\int_0^a f(x)\,dx = af(\frac{a}{2})$
   (b) Hence, or otherwise, evaluate $\displaystyle\int_0^{2\pi} \frac{1}{e^{\sin x} + 1}\,dx.$

6. Let $f(x)$ be a periodic function with period T, prove that, for any real number a and nonzero real number k,
$$\int_a^{a+T/k} f(kx)\,dx = \int_0^{T/k} f(kx)\,dx.$$

7. By considering a suitable definite integral, evaluate the following limits:
   (a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right).$

(b) $\lim_{n\to\infty} \dfrac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$, where $p > 0$.

(c) $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} \left(\dfrac{1}{\sqrt{1+n}} + \dfrac{1}{\sqrt{2+n}} + \cdots + \dfrac{1}{\sqrt{n+n}} \right)$.

8. (a) Show that

$$\int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1.$$

(b) Let $f(x)$ be an increasing function on $[0,1]$. Show that

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k-1}{n}\right) \le \int_0^1 f(x)\,dx \le \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

(c) Hence show that

$$\ln\left[\frac{1}{n}\left(\frac{(2n)!}{2(n!)}\right)^{\frac{1}{n}}\right] \le \ln(\frac{4}{e}) \le \ln\left[\frac{1}{n}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{n}}\right].$$

Deduce that

$$\lim_{n\to\infty} \ln\left[\frac{1}{n}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{n}}\right] = \ln(\frac{4}{e}).$$

9. (a) Show that, if $f(x)$ is a continuous function satisfies $m \le f(x) \le M$ for $a \le x \le b$, where m and M are real numbers, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Hence, prove that, if $p \ge 0$,

$$1^p + 2^p + \cdots + (n-1)^p \le \int_0^n x^p\,dx \le 1^p + 2^p + \cdots + n^p.$$

(b) Let $F(n) = \dfrac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$. Using (a), show that

$$\frac{1}{p+1} \le F(n) \le \frac{1}{p+1} + \frac{1}{n}.$$

Hence, find $\lim_{n\to\infty} F(n)$.